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Exact solution of the $A^{(1)}_{n-1}$ trigonometric vertex model with non-diagonal open boundaries

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ABSTRACT: The $A^{(1)}_{n-1}$ trigonometric vertex model with *generic non-diagonal* boundaries is studied. The double-row transfer matrix of the model is diagonalized by algebraic Bethe ansatz method in terms of the intertwiner and the corresponding face-vertex relation. The eigenvalues and the corresponding Bethe ansatz equations are obtained.

KEYWORDS: Lattice Integrable Models, Bethe Ansatz.

Contents

1. Introduction

Two-dimensional integrable models have traditionally been solved by imposing periodic boundary conditions. For such bulk systems, the quantum Yang-Baxter equation (QYBE)

$$R_{12}(u_1-u_2)R_{13}(u_1-u_3)R_{23}(u_2-u_3)=R_{23}(u_2-u_3)R_{13}(u_1-u_3)R_{12}(u_1-u_2)\,, \quad (1.1)$$

leads to families of commuting row-to-row *transfer matrices* which may be diagonalized by the quantum inverse scattering method (QISM) (or algebraic Bethe ansatz) [1].

Not all boundary conditions are compatible with integrability in the bulk. The bulk integrability is only preserved when one imposes certain boundary conditions. In [2], Sklyanin developed the boundary QISM, which may be used to described integrable systems on a finite interval with independent boundary conditions at each end. This boundary QISM uses the new algebraic structure, the reflection equation (RE) algebra. The solutions to the RE and its dual are called boundary K-matrices which in turn give rise to boundary conditions compatible with the integrability of the bulk model [2]–[4].

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The boundary QISM has been applied to diagonalize the double-row transfer matrices of various integrable models with non-trivial boundary conditions mostly corresponding to the diagonal K-matrices. However, the problem of diagonalizing the double-row transfer matrix for general *non-diagonal* K-matrices has been long-standing for trigonometric integrable models. To our knowledge, the only exception is the spin-$\frac{1}{2}$ XXZ (or $A_1^{(1)}$) model with non-diagonal K-matrices which was solved recently by fusion hierarchy of the transfer matrix with the anisotropy value being the roots of unity [5, 6], the algebraic Bethe ansatz [7, 8] and the coordinate Bethe ansatz [9]. The fundamental difficulty is that the usual highest-weight state which is the pseudo-vacuum (or reference state) for the models with periodic boundary condition or boundary conditions specified by diagonal K-matrices is *no longer* the pseudo-vacuum on which the Bethe ansatz analysis is based.

In a very recent work [10], we constructed a class of non-diagonal solutions to the RE for the trigonometric $A_{n-1}^{(1)}$ vertex model by the intertwiner-matrix approach. The *non-diagonal* K-matrices we found can be expressed in terms of the intertwiner-matrices and diagonal face-type K-matrices. In the present paper, we solve the trigonometric $A_{n-1}^{(1)}$ vertex models with boundary conditions given by the non-diagonal K-matrices in [10]. We construct the pseudo-vacuum and diagonalize the corresponding double-row transfer matrix by the generalized QISM developed in [11].

This paper is organized as follows. In section 2, we introduce our notation and some basic ingredients. In section 3, we introduce the intertwiner-matrix which satisfies the face-vertex correspondence relation between the two R-matrices $R(u)$ and $W(u)$. Through the *magic* intertwiner vectors, in section 4, we transform the model from the original vertex picture into its "face" picture. After succeeding in constructing the pseudo-vacuum state, we apply the algebraic Bethe ansatz method to diagonalize the transfer matrices of the boundary model. Section 5 is for conclusions. Some detailed technical calculations are given in appendices A–C.

2. $A_{n-1}^{(1)}$ trigonometric vertex model and integrable boundary conditions

Let us fix a positive integer n ($n \geq 2$) and a generic complex number η, and $R(u) \in End(\mathbf{C}^n \otimes \mathbf{C}^n)$ be the trigonometric solution to the $A_{n-1}^{(1)}$ type QYBE given by [12, 13, 14]

$$R(u) = \sum_{\alpha=1}^{n} R_{\alpha\alpha}^{\alpha\alpha}(u) E_{\alpha\alpha} \otimes E_{\alpha\alpha} + \sum_{\alpha\neq\beta} \left\{ R_{\alpha\beta}^{\alpha\beta}(u) E_{\alpha\alpha} \otimes E_{\beta\beta} + R_{\alpha\beta}^{\beta\alpha}(u) E_{\beta\alpha} \otimes E_{\alpha\beta} \right\}, \quad (2.1)$$

where E_{ij} is the matrix with elements $(E_{ij})_k^l = \delta_{jk}\delta_{il}$. The coefficient functions are

$$R_{\alpha\beta}^{\alpha\beta}(u) = \begin{cases} \frac{\sin(u)}{\sin(u+\eta)}\, e^{-i\eta}\,, & \alpha > \beta\,, \\ 1, & \alpha = \beta\,, \\ \frac{\sin(u)}{\sin(u+\eta)}\, e^{i\eta}\,, & \alpha < \beta\,, \end{cases} \quad (2.2)$$

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$$R^{\beta\alpha}_{\alpha\beta}(u) = \begin{cases} \frac{\sin(\eta)}{\sin(u+\eta)}\, e^{iu}\,, & \alpha > \beta\,, \\ 1, & \alpha = \beta\,, \\ \frac{\sin(\eta)}{\sin(u+\eta)}\, e^{-iu}\,, & \alpha < \beta\,. \end{cases} \tag{2.3}$$

One can check that the R-matrix satisfies the following unitarity, crossing-unitarity and quasi-classical relations:

Unitarity : $$R_{12}(u)R_{21}(-u) = \mathrm{id}\,, \tag{2.4}$$

Crossing-unitarity : $$R^{t_2}_{12}(u)M_2^{-1}R^{t_2}_{21}(-u-n\eta)M_2 = \frac{\sin(u)\sin(u+n\eta)}{\sin(u+\eta)\sin(u+n\eta-\eta)}\,\mathrm{id}\,, \tag{2.5}$$

Quasi-classical property : $$R_{12}(u)|_{\eta\to 0} = \mathrm{id}\,. \tag{2.6}$$

Here $R_{21}(u) = P_{12}R_{12}(u)P_{12}$ with P_{12} being the usual permutation operator and t_i denotes the transposition in the i-th space, and η is the so-called crossing paramter. The crossing matrix M is a diagonal $n \times n$ matrix with elements

$$M_{\alpha\beta} = M_\alpha \delta_{\alpha\beta}\,, \quad M_\alpha = e^{-2i\alpha\eta}, \qquad \alpha = 1, \ldots, n\,. \tag{2.7}$$

Here and below we adopt the standard notation: for any matrix $A \in \mathrm{End}(\mathbf{C}^n)$, A_j is an embedding operator in the tensor space $\mathbf{C}^n \otimes \mathbf{C}^n \otimes \cdots$, which acts as A on the j-th space and as an identity on the other factor spaces; $R_{ij}(u)$ is an embedding operator of R-matrix in the tensor space, which acts as an identity on the factor spaces except for the i-th and j-th ones.

One introduces the "row-to-row" monodromy matrix $T(u)$, which is an $n \times n$ matrix with elements being operators acting on $(\mathbf{C}^n)^{\otimes N}$

$$T(u) = R_{01}(u+z_1)R_{02}(u+z_2)\cdots R_{0N}(u+z_N)\,. \tag{2.8}$$

Here $\{z_i|i = 1, \ldots, N\}$ are arbitrary free complex parameters which are usually called inhomogeneous parameters. With the help of the QYBE (1.1), one can show that $T(u)$ satisfies the so-called "RLL" relation

$$R_{12}(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R_{12}(u-v)\,. \tag{2.9}$$

Integrable open chains can be constructed as follows [2]. Let us introduce the K-matrix $K^-(u)$ which gives rise to an integrable boundary condition on the right boundary. $K^-(u)$ satisfies the RE

$$R_{12}(u_1-u_2)K^-_1(u_1)R_{21}(u_1+u_2)K^-_2(u_2) = K^-_2(u_2)R_{12}(u_1+u_2)K^-_1(u_1)R_{21}(u_1-u_2)\,. \tag{2.10}$$

For models with open boundaries, instead of the standard "row-to-row" monodromy matrix $T(u)$ (2.8), one needs the "double-row" monodromy matrix $\mathbf{T}(u)$

$$\mathbf{T}(u) = T(u)K^-(u)T^{-1}(-u)\,. \tag{2.11}$$

Using (2.9) and (2.10), one can prove that $\mathbf{T}(u)$ satisfies

$$R_{12}(u_1-u_2)\mathbf{T}_1(u_1)R_{21}(u_1+u_2)\mathbf{T}_2(u_2)=\mathbf{T}_2(u_2)R_{12}(u_1+u_2)\mathbf{T}_1(u_1)R_{21}(u_1-u_2)\,. \quad (2.12)$$

In order to construct the *double-row transfer matrices*, besides the RE, one needs another K-matrix $K^+(u)$ which gives integrable boundary condition on the left boundary. The K-matrix $K^+(u)$ satisfies the dual RE [2, 3, 10]

$$\begin{aligned} &R_{12}(u_2-u_1)K_1^+(u_1)\,M_1^{-1}\,R_{21}(-u_1-u_2-n\eta)\,M_1\,K_2^+(u_2)=\\ &\quad=M_1\,K_2^+(u_2)R_{12}(-u_1-u_2-n\eta)\,M_1^{-1}\,K_1^+(u_1)R_{21}(u_2-u_1)\,. \end{aligned} \quad (2.13)$$

Different integrable boundary conditions are described by different solutions $K^-(u)$ ($K^+(u)$) to the (dual) RE [2, 4]. In this paper, we consider the non-diagonal solutions $K^\pm(u)$ obtained in [10], which are respectively given by

$$K^-(u)^s_t=\sum_{i=1}^n k_i(u)\phi^{(s)}_{\lambda,\lambda-\epsilon_i}(u)\bar{\phi}^{(t)}_{\lambda,\lambda-\epsilon_i}(-u)\,, \quad (2.14)$$

$$K^+(u)^s_t=\sum_{i=1}^n \tilde{k}_i(u)\phi^{(s)}_{\lambda',\lambda'-\epsilon_i}(-u)\tilde{\phi}^{(t)}_{\lambda',\lambda'-\epsilon_i}(u)\,. \quad (2.15)$$

Here $\{k_i(u)|i=1,\ldots,n\}$ and $\{\tilde{k}_i(u)|i=1,\ldots,n\}$ are

$$k_j(u)=\begin{cases}1, & 1\le j\le l\,,\\ \frac{\sin(\xi-u)}{\sin(\xi+u)}e^{-2iu}, & l+1\le j\le n\,,\end{cases} \quad (2.16)$$

$$\tilde{k}_j(u)=\begin{cases}e^{-2i(j\eta)}, & 1\le j\le l'\,,\\ \frac{\sin(\bar{\xi}+u+\frac{n}{2}\eta)}{\sin(\bar{\xi}-u-\frac{n}{2}\eta)}e^{2i(u+\frac{n-2j}{2}\eta)}, & l'+1\le j\le n\,,\end{cases} \quad (2.17)$$

where l and l' are positive integers such that $1\le l\le n$, $1\le l'\le n$. In (2.14) and (2.15), ϕ, $\bar{\phi}$ and $\tilde{\phi}$ are intertwiners which will be specified in section 3. The K-matrix $K^-(u)$ (resp. $K^+(u)$) depends on a *discrete* parameter l (resp. l') and *continuous* parameters ξ, $\{\lambda_i\}$ (resp. $\bar{\xi}$, $\{\lambda'_i\}$) and ρ (whose dependence is through the definition of the intertwiner-matrix (3.2) below). Here and throughout we use the convention: associated with the boundary parameters $\{\lambda_i\}$ (resp. $\{\lambda'_i\}$) let us introduce a vector $\lambda=\sum_{i=1}^n\lambda_i\epsilon_i$ (resp. $\lambda'=\sum_{i=1}^n\lambda'_i\epsilon_i$), where $\{\epsilon_i|\,i=1,\ldots,n\}$ is the orthonormal basis of the vector space $\mathbf{C}^n$ such that $\langle\epsilon_i,\epsilon_j\rangle=\delta_{ij}$.

Some remarks are in order. Our bulk R-matrix is different from those used in [15, 16, 17] (see (2.2)). For $n=3$ case, after a similarity transformation by a spectral-independent *diagonal* matrix, our K-matrix $K^-(u)$ has the same number of the boundary parameters as that of [17] and one more boundary parameter than that of [15, 16]. For generic n ($n>3$), our K-matrix $K^-(u)$ has many more boundary parameters than that given in [15].

Let us emphasize that a further restriction

$$\lambda'+\sum_{k=1}^N\epsilon_{j_k}=\lambda\,, \quad (2.18)$$

where $\{j_k|k=1,\ldots,N\}$ are positive integers such that $2 \le j_k \le n$, is necessary for the the application of the algebraic Bethe ansatz method in section 4. Hereafter, we shall consider only the above case.

The *double-row transfer matrix* of the inhomogeneous model associated with the R-matrix (2.1)–(2.3) with open boundary specified by the K-matrices $K^{\pm}(u)$ (2.14)–(2.18) is given by

$$\tau(u) = \mathrm{tr}(K^{+}(u)\mathbf{T}(u)) . \tag{2.19}$$

The commutativity of the transfer matrices

$$[\tau(u), \tau(v)] = 0 , \tag{2.20}$$

follows as a consequence of (1.1), (2.4)–(2.5) and (2.12)–(2.13). This ensures the integrability of the inhomogeneous model with open boundary. The aim of this paper is to find the common eigenvectors and the corresponding eigenvalues of the transfer matrix (2.19) with generic non-diagonal K-matrices $K^{\pm}(u)$ given by (2.14)–(2.18).

3. Intertwining vectors and the associated face-vertex correspondence relations

For a vector $m \in \mathbf{C}^n$, set

$$m_i = \langle m, \epsilon_i \rangle , \quad |m| = \sum_{k=1}^{n} m_k , \qquad i = 1, \ldots, n . \tag{3.1}$$

Let us introduce n intertwining vectors (intertwiners) $\{\phi_{m,m-\epsilon_j}(u)|j = 1,\ldots,n\}$. Each $\phi_{m,m-\epsilon_j}(u)$ is an n-component column vector whose α-th elements are $\{\phi^{(\alpha)}_{m,m-\epsilon_j}(u)\}$. The n intertwiners form an $n \times n$ matrix (in which j and α stand for the column and the row indices respectively), called the intertwiner-matrix, with the non-vanishing matrix elements being

$$\begin{pmatrix} e^{i\eta f_1(m)} & & & & & & e^{i\eta F_n(m)+\rho} e^{2iu} \\ e^{i\eta F_1(m)} & e^{i\eta f_2(m)} & & & & & \\ & e^{i\eta F_2(m)} & \ddots & & & & \\ & & \ddots & e^{i\eta f_j(m)} & & & \\ & & & e^{i\eta F_j(m)} & \ddots & & \\ & & & & \ddots & e^{i\eta f_{n-1}(m)} & \\ & & & & & e^{i\eta F_{n-1}(m)} & e^{i\eta f_n(m)} \end{pmatrix} . \tag{3.2}$$

Here ρ is a complex constant with regard to u and m, and $\{f_i(m)|i = 1,\ldots,n\}$ and $\{F_i(m)|i=1,\ldots,n\}$ are linear functions of m:

$$f_i(m) = \sum_{k=1}^{i-1} m_k - m_i - \frac{1}{2}|m| , \qquad i = 1, \ldots, n , \tag{3.3}$$

$$F_i(m) = \sum_{k=1}^{i} m_k - \frac{1}{2}|m|\,, \qquad i = 1, \ldots, n-1\,, \tag{3.4}$$

$$F_n(m) = -\frac{3}{2}|m|\,. \tag{3.5}$$

From the above intertwiner-matrix, one may derive the following *face-vertex correspondence relation* [10]:

$$R_{12}(u_1 - u_2)\phi_{m,m-\epsilon_i}(u_1) \otimes \phi_{m-\epsilon_i,m-\epsilon_i-\epsilon_j}(u_2) = \\ = \sum_{k,l} W_{ij}^{kl}(u_1 - u_2)\phi_{m-\epsilon_l,m-\epsilon_l-\epsilon_k}(u_1) \otimes \phi_{m,m-\epsilon_l}(u_2)\,. \tag{3.6}$$

Here the non-vanishing elements of $\{W(u)_{ij}^{kl}\}$ are

$$W_{jj}^{jj}(u) = 1\,, \qquad W_{jk}^{jk}(u) = \frac{\sin(u)}{\sin(u+\eta)}\,, \qquad \text{for } j \neq k\,, \tag{3.7}$$

$$W_{jk}^{kj}(u) = \begin{cases} \frac{\sin(\eta)}{\sin(u+\eta)}\, e^{iu}, & j > k\,, \\ \frac{\sin(\eta)}{\sin(u+\eta)}\, e^{-iu}\,, & j < k\,, \end{cases} \qquad \text{for } j \neq k\,. \tag{3.8}$$

Associated with $\{W(u)_{ij}^{kl}\}$, one may introduce "*face*" type R-matrix $W(u)$

$$W(u) = \sum_{i,j,k,l} W_{ij}^{kl}(u) E_{ki} \otimes E_{lj}\,. \tag{3.9}$$

Some remarks are in order. The "face" type R-matrix $W(u)$ does not depend on the face type parameter m, in contrast to the $\mathbf{Z}_n$ elliptic case [18, 19]. It follows that $W(u)$ and $R(u)$ satisfy the same QYBE, i.e. $W(u)$ obeys the usual (vertex type) QYBE rather than the *dynamical* one [20, 11].

Noting that

$$\sum_{i=1}^{n} f_i(m) = \sum_{i=1}^{n} F_i(m) = \sum_{k=1}^{n} \frac{n-2(k+1)}{2} m_k\,, \tag{3.10}$$

one can show that the determinant of the intertwiner matrix (3.2) is

$$\mathrm{Det}\left(\phi_{m,m-\epsilon_j}^{(\alpha)}(u)\right) = e^{i\eta \sum_{k=1}^{n} \frac{n-2(k+1)}{2} m_k}\left(1 - (-1)^n e^{2iu+\rho}\right). \tag{3.11}$$

For a generic $\rho \in \mathbf{C}$ this determinant is non-vanishing and thus the inverse of (3.2) exists. This fact allows us to introduce other types of intertwiners $\bar{\phi}$ and $\tilde{\phi}$ satisfying the following orthogonality conditions:

$$\sum_{\alpha} \bar{\phi}_{m,m-\epsilon_i}^{(\alpha)}(u)\ \phi_{m,m-\epsilon_j}^{(\alpha)}(u) = \delta_{ij}\,, \tag{3.12}$$

$$\sum_{\alpha} \tilde{\phi}_{m+\epsilon_i,m}^{(\alpha)}(u)\ \phi_{m+\epsilon_j,m}^{(\alpha)}(u) = \delta_{ij}\,. \tag{3.13}$$

From these conditions we derive the "completeness" relations:

$$\sum_k \bar{\phi}^{(\alpha)}_{m,m-\epsilon_k}(u)\ \phi^{(\beta)}_{m,m-\epsilon_k}(u) \,=\, \delta_{\alpha\beta}\,, \tag{3.14}$$

$$\sum_k \tilde{\phi}^{(\alpha)}_{m+\epsilon_k,m}(u)\ \phi^{(\beta)}_{m+\epsilon_k,m}(u) \,=\, \delta_{\alpha\beta}\,. \tag{3.15}$$

With the help of (3.12)–(3.15), we obtain, from the face-vertex correspondence relation (3.6),

$$\begin{aligned}&\Big(\tilde{\phi}_{m+\epsilon_k,m}(u_1)\otimes \mathrm{id}\Big)\, R_{12}(u_1-u_2)\,\big(\mathrm{id}\otimes \phi_{m+\epsilon_j,m}(u_2)\big) = \\ &\quad = \sum_{i,l} W^{kl}_{ij}(u_1-u_2)\, \tilde{\phi}_{m+\epsilon_i+\epsilon_j,m+\epsilon_j}(u_1)\otimes \phi_{m+\epsilon_k+\epsilon_l,m+\epsilon_k}(u_2)\,,\end{aligned} \tag{3.16}$$

$$\begin{aligned}&\Big(\tilde{\phi}_{m+\epsilon_k,m}(u_1)\otimes \tilde{\phi}_{m+\epsilon_k+\epsilon_l,m+\epsilon_k}(u_2)\Big)\, R_{12}(u_1-u_2) = \\ &\quad = \sum_{i,j} W^{kl}_{ij}(u_1-u_2)\, \tilde{\phi}_{m+\epsilon_i+\epsilon_j,m+\epsilon_j}(u_1)\otimes \tilde{\phi}_{m+\epsilon_j,m}(u_2)\,,\end{aligned} \tag{3.17}$$

$$\begin{aligned}&\big(\mathrm{id}\otimes \bar{\phi}_{m,m-\epsilon_l}(u_2)\big)\, R_{12}(u_1-u_2)\,\big(\phi_{m,m-\epsilon_i}(u_1)\otimes \mathrm{id}\big) = \\ &\quad = \sum_{k,j} W^{kl}_{ij}(u_1-u_2)\, \phi_{m-\epsilon_l,m-\epsilon_k-\epsilon_l}(u_1)\otimes \bar{\phi}_{m-\epsilon_i,m-\epsilon_i-\epsilon_j}(u_2)\,,\end{aligned} \tag{3.18}$$

$$\begin{aligned}&\big(\bar{\phi}_{m-\epsilon_l,m-\epsilon_k-\epsilon_l}(u_1)\otimes \bar{\phi}_{m,m-\epsilon_l}(u_2)\big)\, R_{12}(u_1-u_2) = \\ &\quad = \sum_{i,j} W^{kl}_{ij}(u_1-u_2)\, \bar{\phi}_{m,m-\epsilon_i}(u_1)\otimes \bar{\phi}_{m-\epsilon_i,m-\epsilon_i-\epsilon_j}(u_2)\,.\end{aligned} \tag{3.19}$$

4. Algebraic Bethe ansatz for the $A^{(1)}_{n-1}$ trigonometric vertex model with non-diagonal open boundaries

In this section, we shall demonstrate that the intertwiners and the face-vertex correspondence relations (3.6)–(3.19) play a fundamental role in the construction of the eigenstates of the $A^{(1)}_{n-1}$ trigonometric vertex model with open boundary condition specified by the K-matrices $K^{\pm}(u)$ given in (2.14)–(2.18). In order to apply the algebraic Bethe ansatz method, we need to transform the fundamental exchange relation (2.12) from the *vertex picture* into its *face picture* so that we can construct the corresponding pseudo-vacuum and the associated Bethe ansatz states.

4.1 K-matrices in the "face" picture

Corresponding to the vertex type K-matrices (2.14) and (2.15), one introduces the following face type K-matrices $\mathcal{K}$ and $\tilde{\mathcal{K}}$, as in [21]

$$\mathcal{K}(\lambda|u)^j_i \,=\, \sum_{\alpha,\beta} \tilde{\phi}^{(\alpha)}_{\lambda-\epsilon_i+\epsilon_j,\ \lambda-\epsilon_i}(u)\, K^-(u)^\alpha_\beta\, \phi^{(\beta)}_{\lambda,\ \lambda-\epsilon_i}(-u)\,, \tag{4.1}$$

$$\tilde{\mathcal{K}}(\lambda'|u)^j_i \,=\, \sum_{\alpha,\beta} \bar{\phi}^{(\alpha)}_{\lambda',\ \lambda'-\epsilon_j}(-u)\, K^+(u)^\alpha_\beta\, \phi^{(\beta)}_{\lambda'-\epsilon_j+\epsilon_i,\ \lambda'-\epsilon_j}(u)\,. \tag{4.2}$$

Through straightforward calculations, we find the $\lambda(\lambda')$-independent face type K-matrices have *diagonal* forms[1]

$$\mathcal{K}(\lambda|u)_i^j = \delta_i^j\, k(u;\xi)_i\,, \quad \tilde{\mathcal{K}}(\lambda'|u)_i^j = \delta_i^j\, \tilde{k}(u)_i\,, \qquad i,j = 1,\ldots,n\,, \tag{4.3}$$

where functions $\{k(u;\xi)_i = k(u)_i\}$ and $\{\tilde{k}(u)_i\}$ are respectively given by (2.16) and (2.17).

Although the K-matrices $K^{\pm}(u)$ given by (2.14) and (2.15) are generally non-diagonal (in the vertex picture), after the face-vertex transformations (4.1) and (4.2), the face type counterparts $\mathcal{K}(\lambda|u)$ and $\tilde{\mathcal{K}}(\lambda'|u)$ become diagonal *simultaneously*. This fact enables us to apply the generalized algebraic Bethe ansatz method developed in [11] for SOS type integrable models to diagonalize the transfer matrix $\tau(u)$ (2.19).

4.2 Exchange relations of double-row monodromy matrix of face type

By means of (3.14), (3.15), (4.2) and (4.3), the transfer matrix $\tau(u)$ (2.19) can be recasted into the following face type form:

$$\begin{aligned}
\tau(u) &= \mathrm{tr}(K^+(u)\mathbf{T}(u)) \\
&= \sum_{\mu,\nu} \mathrm{tr}(K^+(u)\,\phi_{\lambda'-\epsilon_\mu+\epsilon_\nu,\lambda'-\epsilon_\mu}(u)\,\tilde{\phi}_{\lambda'-\epsilon_\mu+\epsilon_\nu,\lambda'-\epsilon_\mu}(u)\,\mathbf{T}(u)\phi_{\lambda',\lambda'-\epsilon_\mu}(-u)\bar{\phi}_{\lambda',\lambda'-\epsilon_\mu}(-u)) \\
&= \sum_{\mu,\nu} \bar{\phi}_{\lambda',\lambda'-\epsilon_\mu}(-u)K^+(u)\,\phi_{\lambda'-\epsilon_\mu+\epsilon_\nu,\lambda'-\epsilon_\mu}(u)\;\tilde{\phi}_{\lambda'-\epsilon_\mu+\epsilon_\nu,\lambda'-\epsilon_\mu}(u)\;\mathbf{T}(u)\phi_{\lambda',\lambda'-\epsilon_\mu}(-u) \\
&= \sum_{\mu,\nu} \tilde{\mathcal{K}}(\lambda'|u)_\nu^\mu\,\mathcal{T}(\lambda'|u)_\mu^\nu = \sum_\mu \tilde{k}(u)_\mu\,\mathcal{T}(\lambda'|u)_\mu^\mu\,.
\end{aligned} \tag{4.4}$$

Here we have introduced the face type double-row monodromy matrix $\mathcal{T}(m|u)$,

$$\begin{aligned}
\mathcal{T}(\lambda'|u)_\mu^\nu &= \mathcal{T}(m|u)_\mu^\nu\,|_{m=\lambda'} = \tilde{\phi}_{m-\epsilon_\mu+\epsilon_\nu,m-\epsilon_\mu}(u)\;\mathbf{T}(u)\,\phi_{m,m-\epsilon_\mu}(-u)\,|_{m=\lambda'} \\
&\equiv \sum_{\alpha,\beta} \tilde{\phi}^{(\beta)}_{\lambda'-\epsilon_\mu+\epsilon_\nu,\lambda'-\epsilon_\mu}(u)\;\mathbf{T}(u)_\alpha^\beta\,\phi^{(\alpha)}_{\lambda',\lambda'-\epsilon_\mu}(-u)\,.
\end{aligned} \tag{4.5}$$

Moreover from (2.12), (3.6) and (3.15) we derive the following exchange relations among $\mathcal{T}(m|u)_\mu^\nu$ (see appendix A for details):

$$\begin{aligned}
&\sum_{i_1,i_2}\sum_{j_1,j_2} W_{i_1\,j_1}^{i_0\,j_0}(u_1-u_2)\,\mathcal{T}(m+\epsilon_{j_1}+\epsilon_{i_2}|u_1)_{i_2}^{i_1} \times W_{j_2\,i_3}^{j_1\,i_2}(u_1+u_2)\,\mathcal{T}(m+\epsilon_{j_3}+\epsilon_{i_3}|u_2)_{j_3}^{j_2} = \\
&\quad= \sum_{i_1,i_2}\sum_{j_1,j_2} \mathcal{T}(m+\epsilon_{j_1}+\epsilon_{i_0}|u_2)_{j_1}^{j_0}\,W_{i_1\,j_2}^{i_0\,j_1}(u_1+u_2)\,\times \\
&\quad\quad \times\mathcal{T}(m+\epsilon_{j_2}+\epsilon_{i_2}|u_1)_{i_2}^{i_1}\,W_{j_3\,i_3}^{j_2\,i_2}(u_1-u_2)\,.
\end{aligned} \tag{4.6}$$

For convenience let us introduce the standard notation:

$$\mathcal{A}(m|u) = \mathcal{T}(m|u)_1^1\,,\ \mathcal{B}_i(m|u) = \mathcal{T}(m|u)_i^1\,,\ \mathcal{C}_i(m|u) = \mathcal{T}(m|u)_1^i\,, \quad i = 2,\ldots,n\,, \tag{4.7}$$

$$\mathcal{D}_i^j(m|u) = \mathcal{T}(m|u)_i^j - \delta_i^j W_{1\,j}^{j\,1}(2u)\,\mathcal{A}(m|u)\,, \quad i,j = 2,\ldots,n\,. \tag{4.8}$$

[1]The spectral parameter u and the boundary parameter ξ of the reduced double-row monodromy matrices constructed from $\mathcal{K}(\lambda|u)$ will be shifted in each step of the nested Bethe ansatz procedure [11]. Therefore, it is convenient to specify the dependence on the boundary parameter ξ of $\mathcal{K}(\lambda|u)$, in terms of $k(u;\xi)_i$, in addition to the spectral parameter u.

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From (4.6), after some tedious calculation, we find the commutation relations among $\mathcal{A}(m|u)$, $\mathcal{D}(m|u)$ and $\mathcal{B}(m|u)$ (see appendix B for details). Here we give those which are relevant for our purpose

$$\begin{aligned}
&\mathcal{A}(m|u)\mathcal{B}_j(m+\epsilon_j-\epsilon_1|v) = \\
&\quad = \frac{\sin(u+v)\sin(u-v-\eta)}{\sin(u+v+\eta)\sin(u-v)}\,\mathcal{B}_j(m+\epsilon_j-\epsilon_1|v)\mathcal{A}(m+\epsilon_j-\epsilon_1|u) + \\
&\qquad + \frac{\sin(\eta)\sin(2v)e^{i(u-v)}}{\sin(u-v)\sin(2v+\eta)}\,\mathcal{B}_j(m+\epsilon_j-\epsilon_1|u)\mathcal{A}(m+\epsilon_j-\epsilon_1|v) - \\
&\qquad - \frac{\sin(\eta)e^{i(u+v)}}{\sin(u+v+\eta)}\sum_{\alpha=2}^{n}\mathcal{B}_\alpha(m+\epsilon_\alpha-\epsilon_1|u)\mathcal{D}_j^\alpha(m+\epsilon_j-\epsilon_1|v)\,, \qquad (4.9)
\end{aligned}$$

$$\begin{aligned}
&\mathcal{D}_a^k(m|u)\mathcal{B}_j(m+\epsilon_j-\epsilon_1|v) = \\
&\quad = \frac{\sin(u-v+\eta)\sin(u+v+2\eta)}{\sin(u-v)\sin(u+v+\eta)}\times \\
&\qquad \times\Bigg\{\sum_{\alpha_1,\alpha_2,\beta_1,\beta_2=2}^{n} W_{\alpha_2\ \beta_1}^{k\ \beta_2}(u+v+\eta)W_{j\ a}^{\beta_1\,\alpha_1}(u-v)\times \\
&\qquad\qquad \times\mathcal{B}_{\beta_2}(m+\epsilon_k+\epsilon_{\beta_2}-\epsilon_a-\epsilon_1|v)\mathcal{D}_{\alpha_1}^{\alpha_2}(m+\epsilon_j-\epsilon_1|u)\Bigg\} - \\
&\qquad - \frac{\sin(\eta)\sin(2u+2\eta)e^{-i(u-v)}}{\sin(u-v)\sin(2u+\eta)}\times \\
&\qquad \times\left\{\sum_{\alpha,\beta=2}^{n} W_{\alpha\ a}^{k\,\beta}(2u+\eta)\mathcal{B}_\beta(m+\epsilon_\alpha-\epsilon_1|u)\mathcal{D}_j^\alpha(m+\epsilon_j-\epsilon_1)|v)\right\} + \\
&\qquad + \frac{\sin(\eta)\sin(2v)\sin(2u+2\eta)e^{-i(u+v)}}{\sin(u+v+\eta)\sin(2v+\eta)\sin(2u+\eta)}\times \\
&\qquad \times\left\{\sum_{\alpha=2}^{n} W_{j\ a}^{k\,\alpha}(2u+\eta)\,\mathcal{B}_\alpha(m+\epsilon_j-\epsilon_1|u)\mathcal{A}(m+\epsilon_j-\epsilon_1|v)\right\}, \qquad (4.10)
\end{aligned}$$

$$\begin{aligned}
&\mathcal{B}_i(m+\epsilon_i-\epsilon_1|u)\mathcal{B}_j(m+\epsilon_i+\epsilon_j-2\epsilon_1|v) = \\
&\quad = \sum_{\alpha,\beta=2}^{n} W_{j\ i}^{\beta\,\alpha}(u-v)\,\mathcal{B}_\beta(m+\epsilon_\beta-\epsilon_1|v)\,\mathcal{B}_\alpha(m+\epsilon_\alpha+\epsilon_\beta-2\epsilon_1|u)\,. \qquad (4.11)
\end{aligned}$$

4.3 Pseudo-vacuum state

The algebraic Bethe ansatz requires, in addition to the relevant commutation relations (4.9)–(4.11), a pseudo-vacuum state which is the common eigenstate of the operators $\mathcal{A}$, $\mathcal{D}_i^i$ and is annihilated by the operators $\mathcal{C}_i$. In contrast to the models with *diagonal* $K^\pm(u)$ [2, 22], for models with *non-diagonal* K-matrices, the usual highest-weight state

$$\begin{pmatrix}1\\0\\\vdots\end{pmatrix}\otimes\cdots\otimes\begin{pmatrix}1\\0\\\vdots\end{pmatrix}, \qquad (4.12)$$

is no longer the pseudo-vacuum state. However, after the face-vertex transformations (4.1) and (4.2), the face type K-matrices $\mathcal{K}(\lambda|u)$ and $\tilde{\mathcal{K}}(\lambda|u)$ *simultaneously* become diagonal. This suggests that one can translate the $A^{(1)}_{n-1}$ trigonometric vertex model with non-diagonal K-matrices (2.14) and (2.15) into the corresponding "face" type model with *diagonal* K-matrices $\mathcal{K}(\lambda|u)$ and $\tilde{\mathcal{K}}(\lambda|u)$ given by (4.1) and (4.2) respectively. Then one can construct the pseudo-vacuum in the "face" picture and use the generalized algebraic Bethe ansatz method [11] to diagonalize the transfer matrix. Such a method has already been successfully used to diagonalize the transfer matrix of $A^{(1)}_1$ trigonometric vertex model (XXZ model) with non-diagonal K-matrices [8]. In this paper we shall extend the construction to the generic $A^{(1)}_{n-1}$ case.

Before introducing the pseudo-vacuum state, let us introduce a generic state in the quantum space by the column vectors of the intertwiner-matrix (3.2)

$$\begin{aligned}|i_1,\ldots,i_N\rangle^m_{m_0} &= \phi^N_{m_0,m_0-\epsilon_{i_N}}(-z_N)\phi^{N-1}_{m_0-\epsilon_{i_N},m_0-\epsilon_{i_N}-\epsilon_{i_{N-1}}}(-z_{N-1})\times\cdots\\ &\quad\times\phi^1_{m_0-\sum_{k=2}^N\epsilon_{i_k},m_0-\sum_{k=1}^N\epsilon_{i_k}}(-z_1)\,,\end{aligned}\tag{4.13}$$

where the vectors $m_0, m\in\mathbf{C}^n$ and $m=m_0-\sum_{k=1}^N\epsilon_{i_k}$, the vector $\phi^k = id\otimes id\cdots\otimes\overset{k-th}{\phi}$ $\otimes id\cdots$.

Now let us evaluate the action of the face type monodromy matrix $\mathcal{T}(m|u)$ (4.5) on the state (4.13). Using the definition of the vertex type double-row monodromy matrix $\mathbf{T}(u)$ (2.11) and relations (3.14)–(3.15), we can further write the face type $\mathcal{T}(m|u)$ in the following form

$$\begin{aligned}\mathcal{T}(m|u)^j_i &= \tilde{\phi}_{m-\epsilon_i+\epsilon_j,m-\epsilon_i}(u)T(u)K^-(u)T^{-1}(-u)\phi_{m,m-\epsilon_i}(-u)\\ &\equiv \mathcal{T}(m,m_0|u)^j_i\\ &= \sum_{\mu,\nu}\tilde{\phi}_{m-\epsilon_i+\epsilon_j,m-\epsilon_i}(u)T(u)\phi_{m_0-\epsilon_\nu+\epsilon_\mu,m_0-\epsilon_\nu}(u)\tilde{\phi}_{m_0-\epsilon_\nu+\epsilon_\mu,m_0-\epsilon_\nu}(u)K^-(u)\times\\ &\quad\times\phi_{m_0,m_0-\epsilon_\nu}(-u)\bar{\phi}_{m_0,m_0-\epsilon_\nu}(-u)T^{-1}(-u)\phi_{m,m-\epsilon_i}(-u)\\ &= \sum_{\mu,\nu}T(m-\epsilon_i,m_0-\epsilon_\nu|u)^j_\mu\mathcal{K}(m_0|u)^\mu_\nu S(m,m_0|u)^\nu_i\,.\end{aligned}\tag{4.14}$$

Here we have introduced

$$T(m,m_0|u)^j_\mu = \tilde{\phi}_{m+\epsilon_j,m}(u)T(u)\phi_{m_0+\epsilon_\mu,m_0}(u)\,,\tag{4.15}$$

$$S(m,m_0|u)^\mu_i = \bar{\phi}_{m_0,m_0-\epsilon_\mu}(-u)T^{-1}(-u)\phi_{m,m-\epsilon_i}(-u)\,,\tag{4.16}$$

$$\mathcal{K}(m_0|u)^j_i = \tilde{\phi}_{m_0-\epsilon_i+\epsilon_j,m_0-\epsilon_i}(u)K^-(u)\phi_{m_0,m_0-\epsilon_i}(-u)\,.\tag{4.17}$$

We can evaluate the action of the operator $T(m,m_0|u)^j_\mu$ on the state $|i_1,\ldots,i_N\rangle^m_{m_0}$ from the definition (2.8) and the face-vertex correspondence relation (3.6)

$$\begin{aligned}&T(m,m_0|u)^j_\mu|i_1,\ldots,i_N\rangle^m_{m_0} =\\ &\quad= \tilde{\phi}^0_{m+\epsilon_j,m}(u)R_{01}(u+z_1)\phi^1_{m_0-\sum_{k=2}^N\epsilon_{i_k},m_0-\sum_{k=1}^N\epsilon_{i_k}}(-z_1)\times\cdots\\ &\quad\times R_{0N}(u+z_N)\phi^N_{m_0,m_0-\epsilon_{i_N}}(-z_N)\phi^0_{m_0+\epsilon_\mu,m_0}(u)\end{aligned}$$

$$= \sum_{\beta_1, i'_N} \tilde{\phi}^0_{m+\epsilon_j, m}(u) R_{01}(u+z_1) \phi^1_{m_0 - \sum_{k=2}^N \epsilon_{i_k}, m_0 - \sum_{k=1}^N \epsilon_{i_k}}(-z_1) \times \cdots$$

$$\times R_{0\,N-1}(u+z_{N-1}) \phi^0_{m_0+\epsilon_\mu - \epsilon_{i'_N}, m_0 - \epsilon_{i_N}}(u) \phi^{N-1}_{m_0 - \epsilon_{i_N}, m_0 - \epsilon_{i_{N-1}} - \epsilon_{i_N}}(-z_{N-1}) \times$$

$$\times W^{\beta_1\, i'_N}_{\mu\, i_N}(u+z_N) \phi^N_{m_0+\epsilon_\mu, m_0+\epsilon_\mu - \epsilon_{i'_N}}(-z_N)$$

$$\vdots \qquad (4.18)$$

$$= W^{j\ i'_1}_{\beta_{N-1}\, i_1}(u+z_1) W^{\beta_{N-1}\, i'_2}_{\beta_{N-2}\, i_2}(u+z_2) \times \cdots \times W^{\beta_1\ i'_N}_{\mu\ \ i_N}(u+z_N)\, |i'_1, \ldots, i'_N\rangle^{m+\epsilon_j}_{m_0+\epsilon_\mu}\,.$$

In the last equation the repeated indices imply summation over $1, 2, \ldots, n$. Noting the unitarity of the R-matrix (2.4), $T^{-1}(-u)$ can be written

$$T_0^{-1}(-u) = R_{N\,0}(u - z_N) \cdots R_{1\,0}(u - z_1)\,. \qquad (4.19)$$

Then by the face-vertex correspondence relation (3.6) we can evaluate the action of the operator $S(m, m_0|u)^\mu_i$ on the state $|i_1, \ldots, i_N\rangle^m_{m_0}$, similar to what we have done for $T(m, m_0|u)$:

$$S(m, m_0|u)^\mu_i |i_1, \ldots, i_N\rangle^m_{m_0} = W^{i'_N\, \mu}_{i_N\, \alpha_{N-1}}(u - z_N) W^{i'_{N-1}\, \alpha_{N-1}}_{i_{N-1}\, \alpha_{N-2}}(u - z_{N-1}) \times \cdots$$
$$\times W^{i'_1\ \alpha_1}_{i_1\ \ i}(u - z_1)\, |i'_1, \ldots, i'_N\rangle^{m-\epsilon_i}_{m_0-\epsilon_\mu}\,. \qquad (4.20)$$

Here the repeated indices are summed over $1, 2, \ldots, n$. Similarly, by the decomposition relation (4.14) and the equations (4.18), (4.20) we obtain the action of $\mathcal{T}(m|u)^j_i$ on the state $|i_1, \ldots, i_N\rangle^m_{m_0}$:

$$\begin{aligned} \mathcal{T}(m|u)^j_i |i_1, \ldots, i_N\rangle^m_{m_0} &\equiv \mathcal{T}(m, m_0|u)^j_i |i_1, \ldots, i_N\rangle^m_{m_0} \\ &= T(m - \epsilon_i, m_0 - \epsilon_\nu|u)^j_\mu \mathcal{K}(m_0|u)^\mu_\nu S(m, m_0|u)^\nu_i |i_1, \ldots, i_N\rangle^m_{m_0} \\ &= W^{j\ i''_1}_{\beta_{N-1}\, i'_1}(u+z_1) W^{\beta_{N-1}\, i''_2}_{\beta_{N-2}\, i'_2}(u+z_2) \times \cdots \\ &\quad \times W^{\beta_1\, i''_N}_{\mu\, i'_N}(u+z_N)\, \mathcal{K}(m_0|u)^\mu_\nu\, W^{i'_N\, \nu}_{i_N\, \alpha_{N-1}}(u - z_N) \times \\ &\quad \times W^{i'_{N-1}\, \alpha_{N-1}}_{i_{N-1}\, \alpha_{N-2}}(u - z_{N-1}) \cdots W^{i'_1\ \alpha_1}_{i_1\ \ i}(u - z_1)\, |i''_1, \ldots, i''_N\rangle^{m-\epsilon_i+\epsilon_j}_{m_0-\epsilon_\nu+\epsilon_\mu}\,. \end{aligned} \qquad (4.21)$$

Here again it is understood that the repeated indices are summed over $1, 2, \ldots, n$.

Specializing the face-type parameters $\{(m_0)_i\}$ to the boundary parameters $\{\lambda_i\}$, i.e. $m = \lambda$, in equation (4.17), then from equation (4.3) the corresponding face type boundary K-matrix $\mathcal{K}(\lambda|u)$ becomes diagonal. This enables us to construct the pseudo-vacuum state of the model and apply the algebraic Bethe ansatz method to diagonalize the *double-row transfer matrices* (2.19) later.

Now, let us construct the pseudo-vacuum state $|\Omega\rangle$:

$$|\Omega\rangle \equiv |\mathrm{vac}\rangle^{\lambda - N\epsilon_1}_{\lambda} = |1, \ldots, 1\rangle^{\lambda - N\epsilon_1}_{\lambda}\,, \qquad (4.22)$$

where λ is related to the boundary parameters $\{\lambda_i\}$ of the boundary K-matrix $K^-(u)$ in (2.14). Then from equations (4.18) and (4.20) we find that the actions of the operators

$T(\lambda - N\epsilon_1, \lambda|u)$ given by (4.15) and $S(\lambda - N\epsilon_1, \lambda|u)$ given by (4.16) on the pseudo-vacuum state (4.22) are given by

$$T(\lambda - N\epsilon_1, \lambda|u)_1^1 |\text{vac}\rangle_\lambda^{\lambda - N\epsilon_1} = |\text{vac}\rangle_{\lambda+\epsilon_1}^{\lambda - N\epsilon_1 + \epsilon_1}\,, \tag{4.23}$$

$$T(\lambda - N\epsilon_1, \lambda|u)_1^i |\text{vac}\rangle_\lambda^{\lambda - N\epsilon_1} = 0\,, \quad i = 2, \ldots, n\,, \tag{4.24}$$

$$T(\lambda - N\epsilon_1, \lambda|u)_j^i |\text{vac}\rangle_\lambda^{\lambda - N\epsilon_1} = \delta_j^i \prod_{k=1}^N W_{j1}^{j1}(u + z_k)\, |\text{vac}\rangle_{\lambda+\epsilon_j}^{\lambda - N\epsilon_1 + \epsilon_j}\,, \quad i,j = 2, \ldots, n\,, \tag{4.25}$$

$$S(\lambda - N\epsilon_1, \lambda|u)_1^1 |\text{vac}\rangle_\lambda^{\lambda - N\epsilon_1} = |\text{vac}\rangle_{\lambda-\epsilon_1}^{\lambda - N\epsilon_1 - \epsilon_1}\,, \tag{4.26}$$

$$S(\lambda - N\epsilon_1, \lambda|u)_1^i |\text{vac}\rangle_\lambda^{\lambda - N\epsilon_1} = 0\,, \quad i = 2, \ldots, n\,, \tag{4.27}$$

$$S(\lambda - N\epsilon_1, \lambda|u)_j^i |\text{vac}\rangle_\lambda^{\lambda - N\epsilon_1} = \delta_j^i \prod_{k=1}^N W_{1j}^{1j}(u - z_k)\, |\text{vac}\rangle_{\lambda-\epsilon_j}^{\lambda - N\epsilon_1 - \epsilon_j}\,, \quad i,j = 2, \ldots, n\,. \tag{4.28}$$

Noting that the diagonal form of $\mathcal{K}(\lambda|u)$ (4.3) and the above equations, we derive

$$\mathcal{T}(\lambda - N\epsilon_1, \lambda|u)_1^1 |\text{vac}\rangle_\lambda^{\lambda - N\epsilon_1} = k(u;\xi)_1 |\text{vac}\rangle_\lambda^{\lambda - N\epsilon_1}\,, \tag{4.29}$$

$$\mathcal{T}(\lambda - N\epsilon_1, \lambda|u)_1^i |\text{vac}\rangle_\lambda^{\lambda - N\epsilon_1} = 0\,, \qquad i = 2, \ldots, n\,. \tag{4.30}$$

Moreover, after a tedious calculation, we have (see appendix C for details)

$$\begin{aligned}
&\mathcal{T}(\lambda - N\epsilon_1, \lambda|u)_j^i |\text{vac}\rangle_\lambda^{\lambda - N\epsilon_1} = \\
&\quad = \delta_j^i \Bigg\{ W_{1j}^{j1}(2u) k(u;\xi)_1 \left(1 - \prod_{k=1}^N W_{1j}^{1j}(u - z_k) W_{j1}^{j1}(u + z_k) \right) + \\
&\qquad\quad + k(u;\xi)_j \prod_{k=1}^N W_{1j}^{1j}(u - z_k) W_{j1}^{j1}(u + z_k) \Bigg\} |\text{vac}\rangle_\lambda^{\lambda - N\epsilon_1}\,, \quad i,j = 2, \ldots, n\,.
\end{aligned} \tag{4.31}$$

Keeping the definition of operators $\mathcal{A}$ (4.7) and $\mathcal{D}_j^i$ (4.8) in mind, and using the relations (4.29)–(4.31), we find that the pseudo-vacuum state given by (4.22) satisfies the following equations

$$\mathcal{A}(\lambda - N\epsilon_1|u)|\Omega\rangle = k(u;\xi)_1 |\Omega\rangle\,, \tag{4.32}$$

$$\begin{aligned}
\mathcal{D}_j^a(\lambda - N\epsilon_1|u)|\Omega\rangle &= \delta_j^a \frac{\sin(2u) e^{i\eta}}{\sin(2u + \eta)} k\left(u + \frac{\eta}{2}; \xi - \frac{\eta}{2} \right)_j \\
&\quad \times \left\{ \prod_{k=1}^N \frac{\sin(u+z_k)\sin(u-z_k)}{\sin(u+z_k+\eta)\sin(u - z_k + \eta)} \right\} |\Omega\rangle\,, \quad a,j = 2, \ldots, n\,,
\end{aligned} \tag{4.33}$$

$$\mathcal{C}_i(\lambda - N\epsilon_1|u)|\Omega\rangle = 0\,, \quad i = 2, \ldots, n\,, \tag{4.34}$$

$$\mathcal{B}_i(\lambda - N\epsilon_1|u)|\Omega\rangle \neq 0\,, \quad i = 2, \ldots, n\,, \tag{4.35}$$

as required. In deriving the equation (4.33), we have used the following equation

$$k(u;\xi)_j - k(u;\xi)_1 W_{1j}^{j1}(2u) = \frac{\sin(2u) e^{i\eta}}{\sin(2u + \eta)} k\left(u + \frac{\eta}{2}; \xi - \frac{\eta}{2} \right)_j\,. \tag{4.36}$$

Therefore, we have constructed the pseudo-vacuum state $|\Omega\rangle$ which is the common eigenstate of the operators $\mathcal{A}$, $\mathcal{D}_i^i$, $i = 2, \ldots, n$, and is annihilated by the operators $\mathcal{C}_i$, $i = 2, \ldots, n$. The operators $\mathcal{B}_i$, $i = 2, \ldots, n$, will play the role of creation operators used to generate the Bethe ansatz states.

4.4 Nested Bethe ansatz

Having derived the relevant commutation relations (4.9)–(4.11) and constructed the pseudo-vacuum state (4.22), we now apply the generalized algebraic Bethe ansatz method developed in [11] to solve the eigenvalue problem for the transfer matrices (2.19) of the $A_{n-1}^{(1)}$ trigonometric vertex model with open boundary condition specified by the K-matrices $K^{\pm}(u)$ given in (2.14)–(2.18).

For convenience, let us introduce a set of non-negative integers $\{N_i | i = 1, \ldots, n-1\}$ with $N_1 = N$ and complex parameters $\{v_k^{(i)} |\ k = 1, 2, \ldots, N_{i+1},\ i = 0, 1, \ldots, n-2\}$. As in the usual nested Bethe ansatz method [23, 24, 22, 20, 11, 25], the parameters $\{v_k^{(i)}\}$ will be used to specify the eigenvectors of the corresponding reduced transfer matrices. They will be constrained later by the Bethe ansatz equations. For convenience, we adopt the following convention:

$$v_k = v_k^{(0)}, \qquad k = 1, 2, \ldots, N. \tag{4.37}$$

We will seek the common eigenvectors (i.e. the so-called Bethe states) of the transfer matrix in the form

$$\begin{aligned} |v_1, \ldots, v_N\rangle = \sum_{i_1, \ldots, i_N = 2}^{n} & F^{i_1, i_2, \ldots, i_N}\, \mathcal{B}_{i_1}(\lambda' + \epsilon_{i_1} - \epsilon_1 | v_1)\, \mathcal{B}_{i_2}(\lambda' + \epsilon_{i_1} + \epsilon_{i_2} - 2\epsilon_1 | v_2) \times \cdots \\ & \times \mathcal{B}_{i_{N-1}}\left(\lambda' + \sum_{k=1}^{N-1} \epsilon_{i_k} - (N-1)\epsilon_1 | v_{N-1}\right) \times \\ & \times \mathcal{B}_{i_N}\left(\lambda' + \sum_{k=1}^{N} \epsilon_{i_k} - N\epsilon_1 | v_N\right) |\Omega\rangle\,. \end{aligned} \tag{4.38}$$

The summing indices in the above equation should obey the following restriction:

$$\lambda' + \sum_{k=1}^{N} \epsilon_{i_k} = \lambda\,, \tag{4.39}$$

where λ' and λ are the boundary parameters which satisfy the restriction (2.18). The condition (4.39) leads to

$$\begin{aligned} |v_1, \ldots, v_N\rangle = \sum_{i_1, \ldots, i_N = 2}^{n} & F^{i_1, i_2, \ldots, i_N}\, \mathcal{B}_{i_1}(\lambda' + \epsilon_{i_1} - \epsilon_1 | v_1)\, \mathcal{B}_{i_2}(\lambda' + \epsilon_{i_1} + \epsilon_{i_2} - 2\epsilon_1 | v_2) \times \cdots \\ & \times \mathcal{B}_{i_{N-1}}\left(\lambda' + \sum_{k=1}^{N-1} \epsilon_{i_k} - (N-1)\epsilon_1 | v_{N-1}\right) \times \\ & \times \mathcal{B}_{i_N}(\lambda - N\epsilon_1 | v_N) |\Omega\rangle\,. \end{aligned} \tag{4.40}$$

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With the help of (4.4), (4.7) and (4.8) we rewrite the transfer matrix (2.19) in terms of the operators $\mathcal{A}$ and $\mathcal{D}_i^i$

$$\begin{aligned}
\tau(u) &= \sum_{\nu=1}^{n} \tilde{k}(u)_\mu \, \mathcal{T}(\lambda'|u)_\mu^\mu \\
&= \tilde{k}(u)_1 \, \mathcal{A}(\lambda'|u) + \sum_{i=2}^{n} \tilde{k}(u)_i \, \mathcal{T}(\lambda'|u)_i^i \\
&= \tilde{k}(u)_1 \, \mathcal{A}(\lambda'|u) + \sum_{i=2}^{n} \tilde{k}(u)_i \, W_{1\,i}^{i1}(2u) \, \mathcal{A}(\lambda'|u) + \\
&\quad + \sum_{i=2}^{n} \tilde{k}(u)_i \left(\mathcal{T}(\lambda'|u)_i^i - W_{1\,i}^{i1}(2u)\mathcal{A}(\lambda'|u)\right) \\
&= \sum_{i=1}^{n} \tilde{k}(u)_i W_{1\,i}^{i1}(2u) \, \mathcal{A}(\lambda'|u) + \\
&\qquad + \sum_{i=2}^{n} \tilde{k}^{(1)}\left(u + \frac{\eta}{2}\right)_i \left(\mathcal{T}(\lambda'|u)_i^i - W_{1\,i}^{i1}(2u)\mathcal{A}(\lambda'|u)\right) \\
&= \alpha^{(1)}(u) \, \mathcal{A}(\lambda'|u) + \sum_{i=2}^{n} \tilde{k}^{(1)}\left(u + \frac{\eta}{2}\right)_i \mathcal{D}(\lambda'|u)_i^i \,.
\end{aligned} \tag{4.41}$$

Here we have used (4.8) and introduced the function $\alpha^{(1)}(u)$,

$$\alpha^{(1)}(u) = \sum_{i=1}^{n} \tilde{k}(u)_i \, W_{1\,i}^{i1}(2u) \,, \tag{4.42}$$

and the reduced K-matrix $\tilde{\mathcal{K}}^{(1)}(\lambda'|u)$ with the elements given by

$$\tilde{\mathcal{K}}^{(1)}(\lambda'|u)_i^j = \delta_i^j \, \tilde{k}^{(1)}(u)_i \,, \qquad i,j = 2,\ldots,n \,, \tag{4.43}$$

$$\tilde{k}^{(1)}(u)_i = \tilde{k}\left(u - \frac{\eta}{2}\right)_i \,, \qquad i = 2,\ldots,n \,. \tag{4.44}$$

To carry out the nested Bethe ansatz process [22, 11, 25] for the $A_{n-1}^{(1)}$ type models with the open boundary conditions, one needs to introduce a set of reduced K-matrices $\{\tilde{\mathcal{K}}^{(b)}(\lambda'|u)|b = 0,\ldots,n-1\}$ [11] which include the original one $\tilde{\mathcal{K}}(\lambda'|u) = \tilde{\mathcal{K}}^{(0)}(\lambda'|u)$ and the ones in (4.43) and (4.44):

$$\tilde{\mathcal{K}}^{(b)}(\lambda'|u)_i^j = \delta_i^j \, \tilde{k}^{(b)}(u)_i \,, \qquad i,j = b+1,\ldots,n \,, \qquad b = 0,\ldots,n-1 \,, \tag{4.45}$$

$$\tilde{k}^{(b)}(u)_i = \tilde{k}\left(u - b\frac{\eta}{2}\right)_i \,, \qquad i = b+1,\ldots,n \,, \qquad b = 0,\ldots,n-1 \,. \tag{4.46}$$

Moreover we introduce a set of functions $\{\alpha^{(b)}(u)|b = 1,\ldots,n-1\}$ (including the one in (4.42)) related to the reduced K-matrices $\tilde{\mathcal{K}}^{(b)}(\lambda'|u)$

$$\alpha^{(b)}(u) = \sum_{i=b}^{n} W_{bi}^{ib}(2u) \, \tilde{k}^{(b-1)}(u)_i \,, \qquad b = 1,\ldots,n \,. \tag{4.47}$$

Carrying out the nested Bethe ansatz, we finally find that, with the coefficients $F^{i_1,i_2,\cdots,i_N}$ in (4.40) properly chosen, the Bethe state $|v_1,\ldots,v_N\rangle$ is the eigenstate of the transfer matrix (2.19),

$$\tau(u)|v_1,\ldots,v_N\rangle = \Lambda(u;\xi,\{v_k\})\,|v_1,\ldots,v_N\rangle\,, \tag{4.48}$$

with eigenvalue given by

$$\begin{aligned}\Lambda(u;\xi,\{v_k\}) &= \alpha^{(1)}(u)k(u;\xi)_1\prod_{k=1}^{N}\frac{\sin(u+v_k)\sin(u-v_k-\eta)}{\sin(u+v_k+\eta)\sin(u-v_k)}+\\&\quad+\frac{\sin(2u)e^{i\eta}}{\sin(2u+\eta)}\left\{\prod_{k=1}^{N}\frac{\sin(u-v_k+\eta)\sigma(u+v_k+2\eta)}{\sin(u-v_k)\sin(u+v_k+\eta)}\times\right.\\&\quad\times\prod_{k=1}^{N}\frac{\sin(u+z_k)\sin(u-z_k)}{\sin(u+z_k+\eta)\sin(u-z_k+\eta)}\times\\&\quad\left.\times\Lambda^{(1)}\left(u+\frac{\eta}{2};\xi-\frac{\eta}{2},\{v_k^{(1)}\}\right)\right\}.\end{aligned} \tag{4.49}$$

The eigenvalues $\{\Lambda^{(j)}(u;\xi,\{v_k^{(j)}\})|j=0,\ldots,n-1\}$ (with $\Lambda(u;\xi,\{v_k\})=\Lambda^{(0)}(u;\xi,\{v_k^{(0)}\})$) of the reduced transfer matrices are given by the following recurrence relations:

$$\begin{aligned}\Lambda^{(j)}(u;\xi^{(j)},\{v_k^{(j)}\}) &= \alpha^{(j+1)}(u)k(u;\xi^{(j)})_{j+1}\prod_{k=1}^{N_{j+1}}\frac{\sin(u+v_k^{(j)})\sin(u-v_k^{(j)}-\eta)}{\sin(u+v_k^{(j)}+\eta)\sin(u-v_k^{(j)})}+\\&\quad+\frac{\sin(2u)e^{i\eta}}{\sin(2u+\eta)}\left\{\prod_{k=1}^{N_{j+1}}\frac{\sin(u-v_k^{(j)}+\eta)\sin(u+v_k^{(j)}+2\eta)}{\sin(u-v_k^{(j)})\sin(u+v_k^{(j)}+\eta)}\times\right.\\&\quad\times\prod_{k=1}^{N_j}\frac{\sin(u+z_k^{(j)})\sin(u-z_k^{(j)})}{\sin(u+z_k^{(j)}+\eta)\sin(u-z_k^{(j)}+\eta)}\times\\&\quad\left.\times\Lambda^{(j+1)}\left(u+\frac{\eta}{2};\xi^{(j)}-\frac{\eta}{2},\{v_k^{(j+1)}\}\right)\right\},\\&\qquad j=1,\ldots,n-2\,,\end{aligned} \tag{4.50}$$

$$\Lambda^{(n-1)}(u;\xi^{(n-1)}) = \tilde{k}^{(n-1)}(u)_n\,k(u;\xi^{(n-1)})_n\,. \tag{4.51}$$

The reduced boundary parameters $\{\xi^{(j)}\}$ and inhomogeneous parameters $\{z_k^{(j)}\}$ are given by

$$\xi^{(j+1)}=\xi^{(j)}-\frac{\eta}{2}\,,\qquad z_k^{(j+1)}=v_k^{(j)}+\frac{\eta}{2}\,,\qquad j=0,\ldots,n-2\,. \tag{4.52}$$

Here we have adopted the convention: $\xi=\xi^{(0)}$, $z_k^{(0)}=z_k$. The complex parameters $\{v_k^{(j)}\}$ satisfy the following Bethe ansatz equations:

$$\alpha^{(1)}(v_s)k(v_s;\xi)_1\frac{\sin(2v_s+\eta)e^{-i\eta}}{\sin(2v_s+2\eta)}\prod_{k\neq s,k=1}^{N_1}\frac{\sin(v_s+v_k)\sin(v_s-v_k-\eta)}{\sin(v_s+v_k+2\eta)\sin(v_s-v_k+\eta)}=$$

$$= \prod_{k=1}^{N} \frac{\sin(v_s + z_k)\sin(v_s - z_k)}{\sin(v_s + z_k + \eta)\sin(v_s - z_k + \eta)} \Lambda^{(1)}\left(v_s + \frac{\eta}{2}; \xi - \frac{\eta}{2}, \{v_k^{(1)}\}\right), \tag{4.53}$$

$$\alpha^{(j+1)}(v_s^{(j)})k(v_s^{(j)};\xi^{(j)})_{j+1} \frac{\sin(2v_s^{(j)} + \eta)e^{-i\eta}}{\sin(2v_s^{(j)} + 2\eta)} \times$$
$$\times \prod_{k\neq s,k=1}^{N_{j+1}} \frac{\sin(v_s^{(j)} + v_k^{(j)})\sin(v_s^{(j)} - v_k^{(j)} - \eta)}{\sin(v_s^{(j)} + v_k^{(j)} + 2\eta)\sin(v_s^{(j)} - v_k^{(j)} + \eta)} =$$
$$= \prod_{k=1}^{N_j} \frac{\sin(v_s^{(j)} + z_k^{(j)})\sin(v_s^{(j)} - z_k^{(j)})}{\sin(v_s^{(j)} + z_k^{(j)} + \eta)\sin(v_s^{(j)} - z_k^{(j)} + \eta)} \Lambda^{(j+1)}\left(v_s^{(j)} + \frac{\eta}{2}; \xi^{(j)} - \frac{\eta}{2}, \{v_k^{(j+1)}\}\right),$$
$$j = 1, \ldots, n-2. \tag{4.54}$$

5. Conclusions

We have studied the $A_{n-1}^{(1)}$ trigonometric vertex model with integrable open boundary condition described by the *generic non-diagonal* boundary K-matrix $K^-(u)$ given in (2.14) and its dual $K^+(u)$ given in (2.15) with restriction (2.18). In addition to the two *discrete* (positive integers) parameters l and l', the total number of the *independent* free boundary parameters: ξ, $\bar{\xi}$, ρ and λ_i, $i = 1, \ldots, n$, is actually $n + 3$. Although the K-matrices given in (2.14) and (2.15) are *non-diagonal* in the vertex picture, they become diagonal *simultaneously* in the "face" picture after the face-vertex transformation given by (4.1)–(4.3). This fact *enables* us to successfully construct the corresponding pseudo-vacuum state $|\Omega\rangle$ (4.22) and apply the algebraic Bethe ansatz method to diagonalize the corresponding *double-row transfer matrices*. The eigenvalues of the transfer matrices and associated Bethe ansatz equations are given by (4.49), (4.50)–(4.52), and (4.53), (4.54). Taking the rational limit of our results with $\{\lim_{\eta\to 0}(\eta\lambda_i)\}$ being kept finite, we recover the results obtained in [26].

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A. The exchange relation of $\mathcal{T}$

The starting point for deriving the exchange relations (4.6) among $\mathcal{T}(m|u)_\mu^\nu$ is the exchange relation (2.12). Multiplying both sides of (2.12) from the right by $\phi_{m+\epsilon_{i_3},m}(-u_1) \otimes \phi_{m+\epsilon_{i_3}+\epsilon_{j_3},m+\epsilon_{i_3}}(-u_2)$, and using the face-vertex correspondence relation (3.6) and the "completeness" relation (3.15), we have, for the l.h.s. of the resulting relation,

$$\begin{aligned}\text{L.H.S.} &= R_{12}(u_1-u_2)\mathbf{T}_1(u_1)R_{21}(u_1+u_2)(\phi_{m+\epsilon_{i_3},m}(-u_1) \otimes \mathbf{T}(u_2)\,\phi_{m+\epsilon_{i_3}+\epsilon_{j_3},m+\epsilon_{i_3}}(-u_2))\\ &= R_{12}(u_1-u_2)\mathbf{T}_1(u_1)R_{21}(u_1+u_2)(\phi_{m+\epsilon_{i_3},m}(-u_1) \otimes 1) \times\\ &\quad \times\Bigg(1 \otimes \Bigg\{ \sum_{j_2} \phi_{m+\epsilon_{i_3}+\epsilon_{j_2},m+\epsilon_{i_3}}(u_2)\tilde{\phi}_{m+\epsilon_{i_3}+\epsilon_{j_2},m+\epsilon_{i_3}}(u_2) \times\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad \times\, \mathbf{T}(u_2)\phi_{m+\epsilon_{i_3}+\epsilon_{j_3},\, m+\epsilon_{i_3}}(-u_2)\Big\}\Big)\\
&= \sum_{j_2} R_{12}(u_1-u_2)\mathbf{T}_1(u_1)R_{21}(u_1+u_2)\,\times\\
&\quad \times \left(\phi_{m+\epsilon_{i_3},\, m}(-u_1)\otimes \phi_{m+\epsilon_{i_3}+\epsilon_{j_2},\, m+\epsilon_{i_3}}(u_2)\right) \mathcal{T}(m+\epsilon_{i_3}+\epsilon_{j_3}|u_2)^{j_2}_{j_3}\\
&= \sum_{i_2}\sum_{j_1,j_2} R_{12}(u_1-u_2)\mathbf{T}_1(u_1)W^{j_1\, i_2}_{j_2\, i_3}(u_1+u_2)\,\times\\
&\quad \times \left(\phi_{m+\epsilon_{i_3}+\epsilon_{j_2},\, m+\epsilon_{j_1}}(-u_1)\otimes \phi_{m+\epsilon_{j_1},\, m}(u_2)\right) \mathcal{T}(m+\epsilon_{i_3}+\epsilon_{j_3}|u_2)^{j_2}_{j_3}\\
&\ \vdots\\
&= \sum_{i_0,j_0} (\phi_{m+\epsilon_{i_0},\, m}(u_1)\otimes \phi_{m+\epsilon_{i_0}+\epsilon_{j_0},\, m+\epsilon_{i_0}}(u_2))\,\times\\
&\qquad \times \Bigg\{ \sum_{i_1,i_2}\sum_{j_1,j_2} W^{i_0\, j_0}_{i_1\, j_1}(u_1-u_2)\mathcal{T}(m+\epsilon_{i_2}+\epsilon_{j_1}|u_1)^{i_1}_{i_2}\,\times\\
&\qquad\quad \times W^{j_1\, i_2}_{j_2\, i_3}(u_1+u_2)\mathcal{T}(m+\epsilon_{i_3}+\epsilon_{j_3}|u_2)^{j_2}_{j_3}\Bigg\}.
\end{aligned}
\tag{A.1}
$$

Similarly for the r.h.s. of the resulting relation, we obtain

$$
\begin{aligned}
\text{R.H.S.} = \sum_{i_0,j_0} &\left(\phi_{m+\epsilon_{i_0},\, m}(u_1)\otimes \phi_{m+\epsilon_{i_0}+\epsilon_{j_0},\, m+\epsilon_{i_0}}(u_2)\right)\times\\
&\times \Bigg\{ \sum_{i_1,i_2}\sum_{j_1,j_2} \mathcal{T}(m+\epsilon_{i_0}+\epsilon_{j_1}|u_2)^{j_0}_{j_1} W^{i_0\, j_1}_{i_1\, j_2}(u_1+u_2)\,\times\\
&\qquad \times \mathcal{T}(m+\epsilon_{i_2}+\epsilon_{j_2}|u_1)^{i_1}_{i_2} W^{j_2\, i_2}_{j_3\, i_3}(u_1-u_2)\Bigg\}.
\end{aligned}
\tag{A.2}
$$

Note that intertwiners are linearly independent, which follows from (3.11). Thus we obtain the exchange relation (4.6) by comparing (A.1) with (A.2).

B. The relevant commutation relations

Let us introduce

$$
D^j_i(m|u) = \mathcal{T}(m|u)^j_i\,, \qquad i,j=2,\ldots,n\,. \tag{B.1}
$$

The starting point for deriving the commutation relations among $\mathcal{A}(m|u)$, $\mathcal{D}^j_i(m|u)$ and $\mathcal{B}_i(m|u)$ $(i,j=2,\ldots,n)$ is the exchange relation (4.6).

For $i_0=j_0=j_3=1,\ i_3=i\neq 1$, we obtain

$$
\begin{aligned}
&\mathcal{A}(m+2\epsilon_1|v)\mathcal{B}_j(m+\epsilon_j+\epsilon_1|u) =\\
&\quad = \frac{\sin(u+v)\sin(u-v+\eta)}{\sin(u+v+\eta)\sin(u-v)}\mathcal{B}_j(m+\epsilon_j+\epsilon_1|u)\mathcal{A}(m+\epsilon_j+\epsilon_1|v) -
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{\sin(\eta)\sin(u+v)e^{-i(u-v)}}{\sin(u-v)\sin(u+v+\eta)} \mathcal{B}_j(m+\epsilon_j+\epsilon_1|v)\mathcal{A}(m+\epsilon_j+\epsilon_1|u) - \\
&- \frac{\sin(\eta)e^{i(u+v)}}{\sin(u+v+\eta)} \sum_{\alpha=2}^{n} \mathcal{B}_\alpha(m+\epsilon_\alpha+\epsilon_1|v) D_j^\alpha(m+\epsilon_j+\epsilon_1|u)\,. \qquad \text{(B.2)}
\end{aligned}
$$

The commutation relation (4.9) is a simple consequence of (B.2), (4.7) and (4.8).

For $i_0 = k \neq 1$, $j_0 = 1$, $i_3 = i \neq 1$ and $j_3 = j \neq 1$, we obtain

$$
\begin{aligned}
&D_a^k(m+\epsilon_a+\epsilon_1|u)\mathcal{B}_j(m+\epsilon_j+\epsilon_a|v) = \\
&\quad = \sum_{\alpha_1,\alpha_2,\beta_1,\beta_2=2}^{n} \frac{W^{k\ \beta_2}_{\alpha_2\,\beta_1}(u+v) W^{\beta_1\,\alpha_1}_{j\ \ a}(u-v)}{W^{k1}_{k1}(u-v)W^{1a}_{1a}(u+v)} \times \\
&\qquad\qquad \times \mathcal{B}_{\beta_2}(m+\epsilon_k+\epsilon_{\beta_2}|v) D^{\alpha_2}_{\alpha_1}(m+\epsilon_a+\epsilon_j|u) + \\
&\quad + \sum_{\alpha=2}^{n} \frac{W^{k\,1}_{1\,k}(u+v) W^{k\,\alpha}_{j\ a}(u-v)}{W^{k1}_{k1}(u-v)W^{1a}_{1a}(u+v)} \mathcal{A}(m+\epsilon_k+\epsilon_1|v)\mathcal{B}_\alpha(m+\epsilon_a+\epsilon_j|u) - \\
&\quad - \sum_{\alpha,\beta=2}^{n} \frac{W^{k\,1}_{1\,k}(u-v) W^{k\,\alpha}_{\beta\ a}(u+v)}{W^{k1}_{k1}(u-v)W^{1a}_{1a}(u+v)} \mathcal{B}_\alpha(m+\epsilon_k+\epsilon_\alpha|u) D^\beta_j(m+\epsilon_a+\epsilon_j|v) - \\
&\quad - \frac{W^{k\,1}_{1\,k}(u-v) W^{k\,1}_{1\,a}(u+v)}{W^{k1}_{k1}(u-v)W^{1a}_{1a}(u+v)} \mathcal{A}(m+\epsilon_k+\epsilon_1|u)\mathcal{B}_j(m+\epsilon_a+\epsilon_j|v)\,. \qquad \text{(B.3)}
\end{aligned}
$$

In order to separate the contribution of $\mathcal{A}$ and D_a^j in the above relations, one needs to introduce the operator $\mathcal{D}_a^j$ as (4.8) (cf. [2]). Then we can derive the commutation relations among $\mathcal{D}_a^k$ and $\mathcal{B}_j$ from (B.3)

$$
\begin{aligned}
&\mathcal{D}_a^k(m|u)\mathcal{B}_j(m+\epsilon_j-\epsilon_1|v) = \\
&\quad = \sum_{\alpha_1,\alpha_2,\beta_1,\beta_2=2}^{n} \frac{W^{k\ \beta_2}_{\alpha_2\,\beta_1}(u+v) W^{\beta_1\,\alpha_1}_{j\ \ a}(u-v)}{W^{k1}_{k1}(u-v)W^{1a}_{1a}(u+v)} \times \\
&\qquad\qquad \times \mathcal{B}_{\beta_2}(m+\epsilon_k+\epsilon_{\beta_2}-\epsilon_a-\epsilon_1|v) \mathcal{D}^{\alpha_2}_{\alpha_1}(m+\epsilon_j-\epsilon_1|u) - \\
&\quad - \sum_{\alpha,\beta=2}^{n} \frac{W^{k\,1}_{1\,k}(u-v) W^{k\,\alpha}_{\beta\ a}(u+v)}{W^{k1}_{k1}(u-v)W^{1a}_{1a}(u+v)} \times \\
&\qquad\qquad \times \mathcal{B}_\alpha(m+\epsilon_\beta-\epsilon_1|u) \mathcal{D}^\beta_j(m+\epsilon_j-\epsilon_1|v) + \\
&\quad + \sum_{\alpha,\beta_1,\beta_2=2}^{n} \frac{W^{k\ \beta_2}_{\alpha\,\beta_1}(u+v) W^{\beta_1\,\alpha}_{j\ \ a}(u-v)}{W^{k1}_{k1}(u-v)W^{1a}_{1a}(u+v)} W^{\alpha 1}_{1\,\alpha}(2u) \times \\
&\qquad\qquad \times \mathcal{B}_{\beta_2}(m+\epsilon_{\beta_2}+\epsilon_k-\epsilon_a-\epsilon_1|v)\mathcal{A}(m+\epsilon_j-\epsilon_1|u) - \\
&\quad - \sum_{\alpha=2}^{n} \frac{W^{k\,1}_{1\,k}(u-v) W^{k\,\alpha}_{j\ a}(u+v)}{W^{k1}_{k1}(u-v)W^{1a}_{1a}(u+v)} W^{j\,1}_{1\,j}(2v) \times \\
&\qquad\qquad \times \mathcal{B}_\alpha(m+\epsilon_\alpha+\epsilon_k-\epsilon_a-\epsilon_1|u)\mathcal{A}(m+\epsilon_j-\epsilon_1|v) + \\
&\quad + \sum_{\alpha=2}^{n} \frac{W^{k\,1}_{1\,k}(u+v) W^{k\,\alpha}_{j\ a}(u-v)}{W^{k1}_{k1}(u-v)W^{1a}_{1a}(u+v)} \mathcal{A}(m+\epsilon_k-\epsilon_a|v)\mathcal{B}_\alpha(m+\epsilon_j-\epsilon_1|u) -
\end{aligned}
$$

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$$-\frac{\sin(u-v+\eta)\sin(u+v+\eta)\sin(\eta)e^{-2iu}}{\sin(u-v)\sin(u+v)\sin(2u+\eta)}\times$$
$$\times\,\delta^k_a\mathcal{A}(m-\epsilon_a+\epsilon_k|u)\mathcal{B}_j(m+\epsilon_j-\epsilon_1|v)\,. \tag{B.4}$$

We have used the following equation

$$\frac{W^{k1}_{1\,k}(u-v)W^{k1}_{1\,k}(u+v)}{W^{k1}_{k1}(u-v)W^{1a}_{1a}(u+v)}+W^{k1}_{1\,k}(2u)=\frac{\sin(u-v+\eta)\sin(u+v+\eta)\sin(\eta)e^{-2iu}}{\sin(u-v)\sin(u+v)\sin(2u+\eta)}\,, \tag{B.5}$$

to derive the last term on the right side of the above equation. After some long tedious calculation, we finally obtain the commutation relation (4.10) from (B.5) by noting the definitions (4.7) and (4.8).

For $i_0=j_0=1$, $i_3=i\neq 1$ and $j_3=j\neq 1$, we obtain

$$\mathcal{B}_i(m+\epsilon_i+\epsilon_1|u)\mathcal{B}_j(m+\epsilon_i+\epsilon_j|v)=$$
$$=\sum_{\beta,\alpha=2}^{n}\frac{W^{\beta\alpha}_{j\,i}(u-v)W^{1\beta}_{1\beta}(u+v)}{W^{11}_{11}(u-v)W^{1i}_{1i}(u+v)}\mathcal{B}_\beta(m+\epsilon_\beta+\epsilon_1|v)\mathcal{B}_\alpha(m+\epsilon_i+\epsilon_j|u)\,, \tag{B.6}$$

which leads to the commutation relation (4.11).

C. The action of $\mathcal{T}^i_j$ on the pseudo-vacuum state

Carrying out the calculation similar to that leading to (4.29) and (4.30), we have

$$\mathcal{T}(\lambda-N\epsilon_1,\lambda|u)^i_j|\mathrm{vac}\rangle^{\lambda-N\epsilon_1}_{\lambda}=$$
$$=k(u;\xi)_1T(\lambda-N\epsilon_1-\epsilon_j,\lambda-\epsilon_1|u)^i_1S(\lambda-N\epsilon_1,\lambda|u)^1_j\,|\mathrm{vac}\rangle^{\lambda-N\epsilon_1}_{\lambda}+$$
$$+\,\delta^i_jk(u;\xi)_j\prod_{k=1}^{N}W^{1j}_{1j}(u-z_k)W^{j1}_{j1}(u+z_k)|\mathrm{vac}\rangle^{\lambda-N\epsilon_1}_{\lambda}\,,\quad i,j=2,\ldots,n\,. \tag{C.1}$$

The first term on the right hand of the above equation is obtained as follows.

From "RLL" relation (2.9), we derive the following exchange relations

$$T_1(u)R_{12}(2u)T_2^{-1}(-u)=T_2^{-1}(-u)R_{12}(2u)T_1(u)\,. \tag{C.2}$$

Multiplying both sides of the above equation from the left by $\tilde{\phi}_{\lambda-N\epsilon_1-\epsilon_i+\epsilon_j,\lambda-N\epsilon_1-\epsilon_i}(u)\otimes\bar{\phi}_{\lambda+\epsilon_1,\lambda}(-u)$ and from the right by $\phi_{\lambda+\epsilon_1,\lambda}(u)\otimes\phi_{\lambda-N\epsilon_1,\lambda-N\epsilon_1-\epsilon_i}(-u)$, we obtain the following exchange relation from the face-vertex correspondence relations (3.6) and (3.16)–(3.19)

$$\sum_{\alpha=1}^{n}W^{\alpha 1}_{1\,\alpha}(2u)T(\lambda-N\epsilon_1-\epsilon_i,\lambda-\epsilon_\alpha|u)^j_\alpha S(\lambda-N\epsilon_1,\lambda|u)^\alpha_i=$$
$$=\sum_{\alpha,\beta=1}^{n}W^{j\,\beta}_{\alpha\,i}(2u)S(\lambda-N\epsilon_1+\epsilon_\alpha,\lambda+\epsilon_1|u)^1_\beta T(\lambda-N\epsilon_1,\lambda|u)^\alpha_1\,. \tag{C.3}$$

Acting both sides on the pseudo-vacuum state $|\text{vac}\rangle_{\lambda}^{\lambda-N\epsilon_1}$, and using the equations (4.23)–(4.28), we obtain

$$T(\lambda-N\epsilon_1-\epsilon_i,\lambda-\epsilon_1|u)_1^j S(\lambda-N\epsilon_1,\lambda|u)_i^1|\text{vac}\rangle_{\lambda}^{\lambda-N\epsilon_1}=$$
$$=\delta_i^j\left\{W_{1j}^{j1}(2u)-W_{1j}^{j1}(2u)\left(\prod_{k=1}^{N}W_{1j}^{1j}(u-z_k)W_{j1}^{j1}(u+z_k)\right)\right\}\times|\text{vac}\rangle_{\lambda}^{\lambda-N\epsilon_1}. \quad \text{(C.4)}$$

The equation (4.31) is a simple consequence of the equations (C.1) and (C.4).

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